Lecture 17

One could ask, "in what direction does f have the greatest rate of change?" Let $\vec{u}$ be a unit vector and θ the angle between ∇f and $\vec{u}$. Then,

$$D_{\vec{u}} f = \nabla f \cdot \vec{u} = |\nabla f||\vec{u}| \cos\theta \underset{|\vec{u}|=1}{=} |\nabla f| \cos\theta.$$

This is largest when $\cos\theta = 1$, i.e, $\theta = 0$. So, the maximum rate of change of f is $|\nabla f|$ and it occurs in the direction of ∇f.

Ex: The temperature in a metal ball is inversely proportional to the distance from the center of the ball, which we assume to be the origin. The temperature at $(1,2,2)$ is 120.

(a) What is the maximum rate of change in temperature at $(1,2,2)$?

(b) What direction is it in?

(c) At any point in the ball, in what direction is the greatest increase in temperature occurring? (i.e., toward which point in the ball?) What does this mean?

Sol: Step 1 is to figure out the temperature function:

$$T(x,y,z) = \frac{\alpha}{\sqrt{x^2+y^2+z^2}}, \quad T(1,2,2) = \frac{\alpha}{3} = 120 \Rightarrow \alpha = 360$$

(a) First we find the gradient of T:

$$\frac{\partial T}{\partial x} = \frac{\partial}{\partial x}\left[360\left(x^2+y^2+z^2\right)^{-1/2}\right] = \left(360\left(x^2+y^2+z^2\right)^{-3/2}\right)\left(\frac{-1}{2}\right)(2x)$$

$$= \frac{-360x}{\left(x^2+y^2+z^2\right)^{3/2}}$$

By symmetry: $\frac{\partial T}{\partial y} = \frac{-360y}{\left(x^2+y^2+z^2\right)^{3/2}}$, $\frac{\partial T}{\partial z} = \frac{-360z}{\left(x^2+y^2+z^2\right)^{3/2}}$

So, $\nabla T = \frac{-360}{\left(x^2+y^2+z^2\right)^{3/2}}\langle x, y, z\rangle$

Thus $\nabla T(1,2,2) = \frac{-360}{(9)^{3/2}}\langle 1,2,2\rangle = \frac{-360}{27}\langle 1,2,2\rangle = \frac{-40}{3}\langle 1,2,2\rangle$

So, the maximum rate of change is $|\nabla T(1,2,2)| = 40$

(b) This occurs in the direction of $\nabla T(1,2,2) = -\frac{40}{3}\langle 1,2,2\rangle$

(c) At any point (x,y,z) the maximum rate of change occurs in the direction of $\nabla T(x,y,z)$. Notice $\nabla T(x,y,z)$ points in the same direction as $-\langle x,y,z\rangle$ and since $-\langle x,y,z\rangle$ connects the point (x,y,z) to the origin, this means ∇T always points towards the origin. This means the center is the hottest point in the ball. This relates to Friday's material. ◇

More on the gradient:

Suppose we have a level surface S given by $F(x,y,z)=k$. To get any vector tangent to S at a point (x_0,y_0,z_0) on S we take a curve $\vec{r}(t)=\langle x(t),y(t),z(t)\rangle$ in S (meaning $F(x(t),y(t),z(t))=k$) passing through (x_0,y_0,z_0) (i.e., there is a t_0 s.t. $\vec{r}(t_0)=\langle x_0,y_0,z_0\rangle$) and take the tangent vector to $\vec{r}$ at t_0, i.e., $\vec{r}\,'(t_0)$.

Now, we know $F(x(t),y(t),z(t))=k$; what happens if we take $\frac{d}{dt}$ of both sides?

$$\frac{dF}{dt}=\frac{\partial F}{\partial x}\frac{dx}{dt}+\frac{\partial F}{\partial y}\frac{dy}{dt}+\frac{\partial F}{\partial z}\frac{dz}{dt}=0$$

components of ∇F

components of $\vec{r}\,'(t)$

This equation thus becomes $\nabla F\cdot\vec{r}\,'(t)=0$, i.e., ∇F is perpendicular to any tangent vector of S, i.e., ∇F is perpendicular to S.

<u>Note</u>: A similar calculation shows ∇G is perpendicular to level curves of G, i.e., curves C given by $G(x,y)=k$.

Using this fact that gradients are perpendicular to level curves, we can estimate the gradient of the function.

Ex: Estimate the gradient at the indicated points given the level curves of a function $f = f(x,y)$. See next page for contour plot.

Sol: See next page for estimated gradients.

The contour plot shows level curves of $f(x,y) = x^2 - y^2$. See mathematica code for a graph containing several gradients plotted. Let's check our guesses. First, $\nabla f(x,y) = \langle 2x, -2y \rangle$.

$\nabla f(0,0) = \langle 0,0 \rangle$: this is a critical point

$\nabla f(1,-1) = \langle 2,2 \rangle$, $\nabla f(-3,0) = \langle -6,0 \rangle$, $\nabla f(0,3) = \langle 0,-6 \rangle$

$\nabla f(-8,6) = \langle -16,-12 \rangle$, $\nabla f(4,-5) = \langle 8,10 \rangle$

$\diamond$

Notice how the gradient is bigger when the level curves are denser. This means the function is increasing faster, so our previous discussion shows that $|\nabla f|$ should be bigger.

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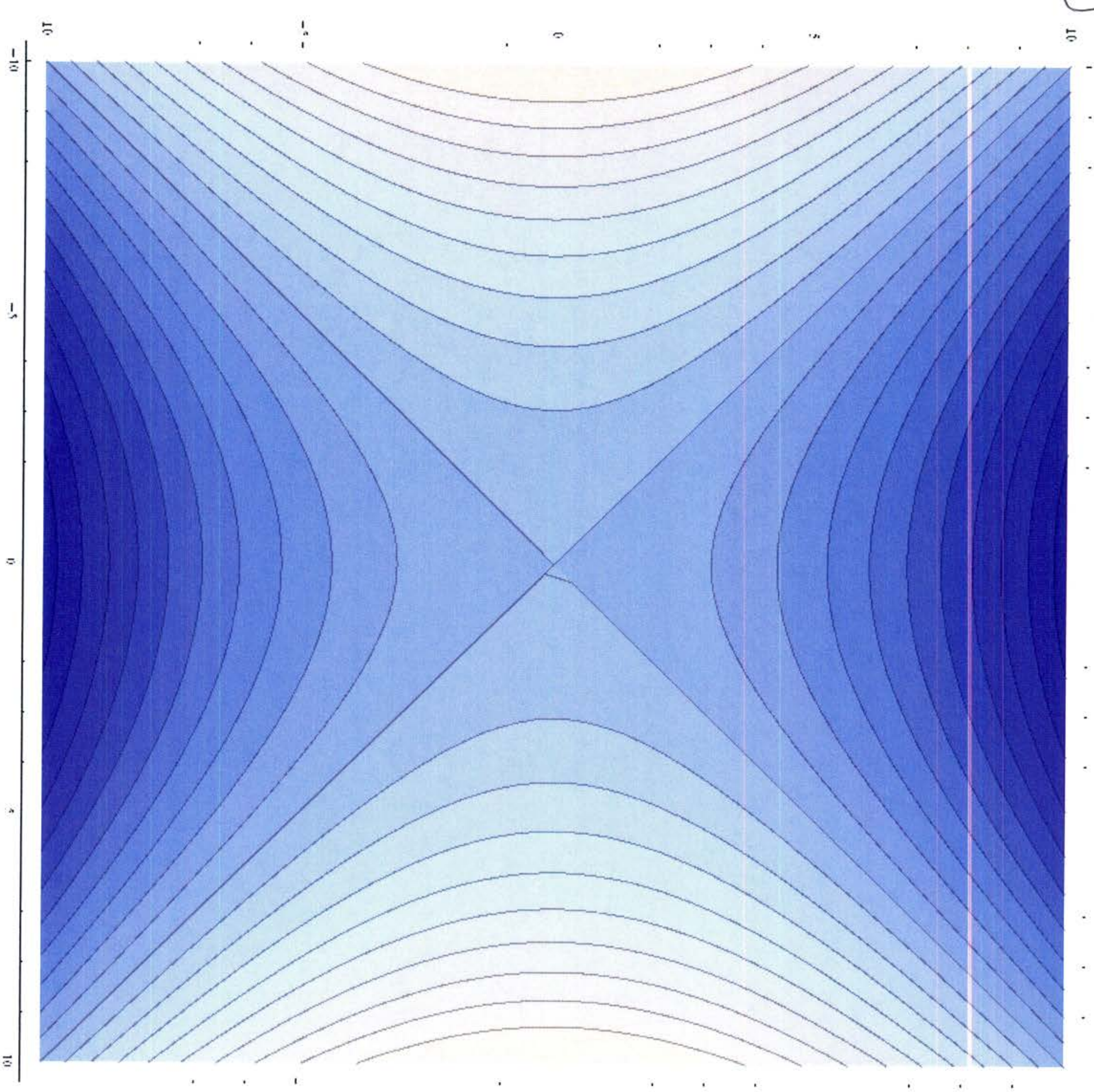
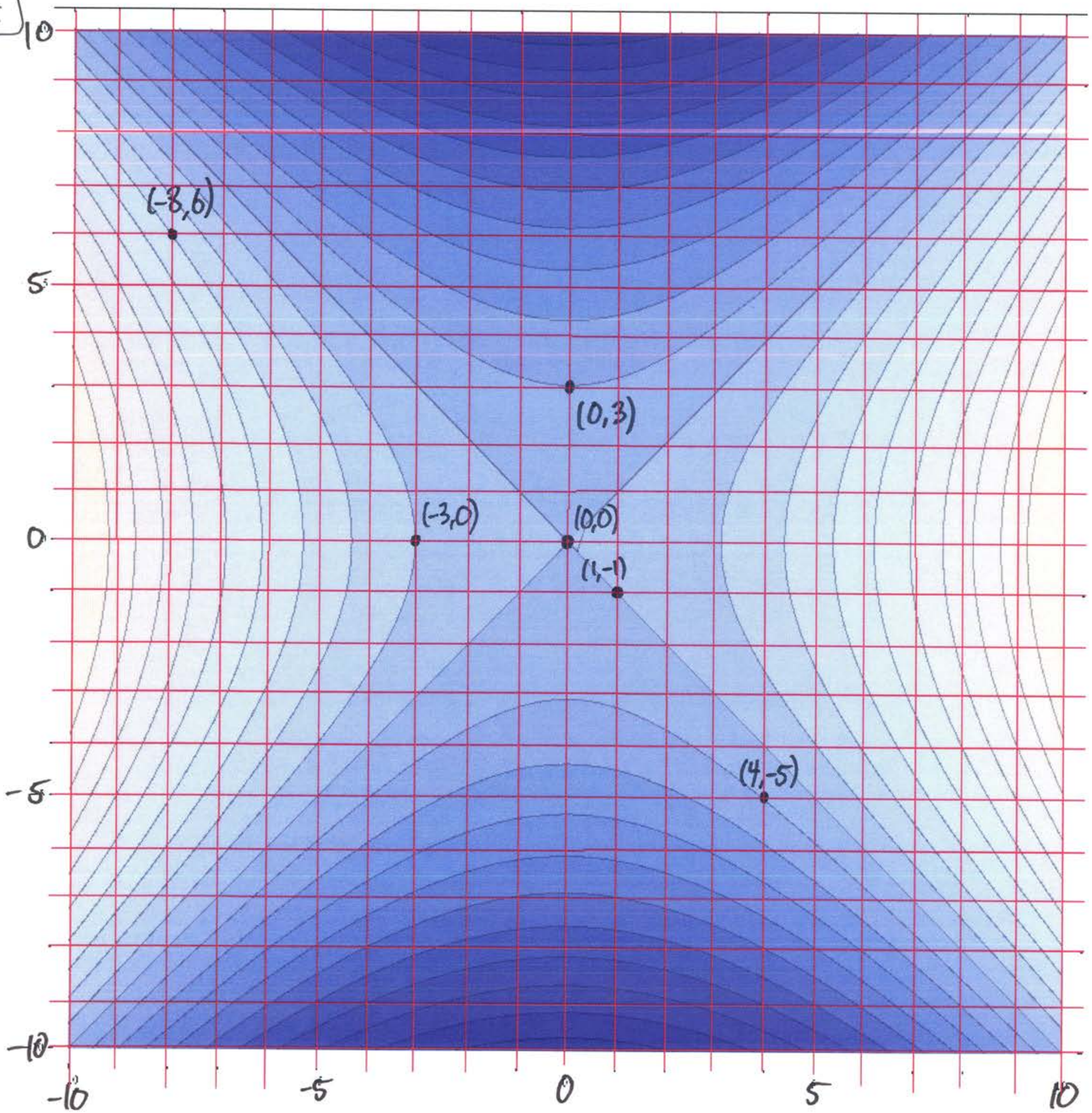
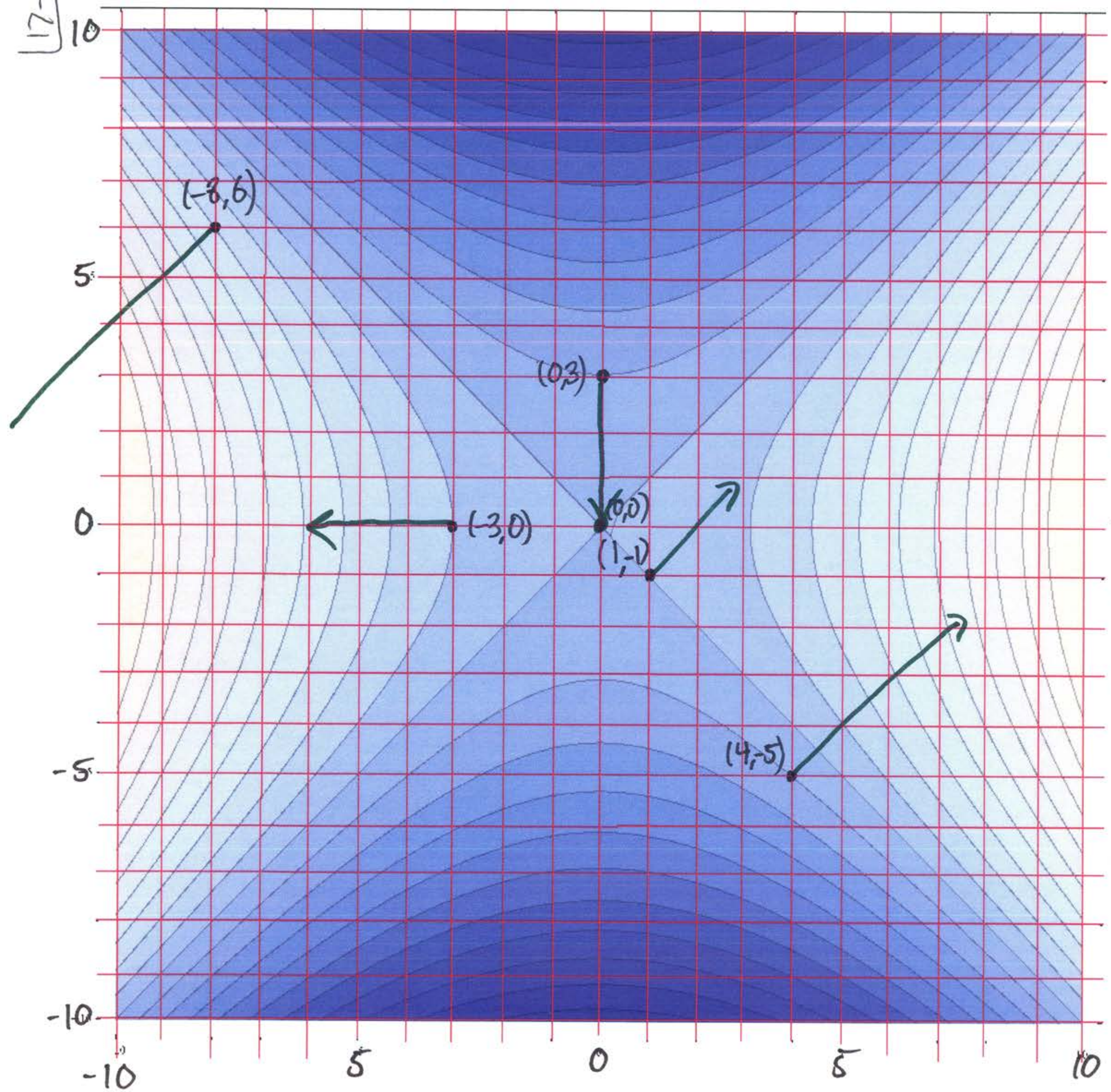




Try it yourself!



More blue indicates decreasing, more tan indicates increasing.

Keep in mind, these are only approximations their lengths are not to scale, but the variance in length among the vectors should be demonstrated.